Quadratic-Argument Approach to the Davey-Stewartson Equations[1]

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Abstract

The Davey-Stewartson equations are used to describe the long time evolution of a three-dimensional packets of surface waves. Assuming that the argument functions are quadratic in spacial variables, we find in this paper various exact solutions modulo the most known symmetry transformations for the Davey-Stewartson equations.

Keywords: the Davey-Stewartson equations, symmetry transformation, exact solution, parameter function.

1 Introduction

The Davey and Stewartson [6] (1974) used the method of multiple scales to derive the following system of nonlinear partial differential equations

$$2iu_t + \epsilon_1 u_{xx} + u_{yy} - 2\epsilon_2 |u|^2 u - 2uv = 0, \tag{1.1}$$

$$v_{xx} - \epsilon_1 (v_{yy} + 2(|u|^2)_{xx}) = 0 \tag{1.2}$$

that describe the long time evolution of a three-dimensional packets of surface waves, where u is a complex-valued function, v is a real valued function and $\epsilon_1, \epsilon_2 = \pm 1$. The equations are called the *Davey-Stewartson I equations* if $\epsilon_1 = 1$, and the *Davey-Stewartson II equations* when $\epsilon_1 = -1$. They were used to study the stability of the uniform Stokes wave train with respect to small disturbance. The soliton solutions of the

[1] 2000 Mathematical Subject Classification: Primary 35C05, 35Q55; Secondary 37K10. PACS: 4710Fg, 47.35.Fg.

[2] Research supported by China NSF 10871193

Davey-Stewartson equations were first studied by Anker and Freeman [2] (1978). Kirby and Dalrymple [7] (1983) obtained oblique envelope solutions of the equations in intermediate water depth. Omote [13] (1988) found infinite-dimensional symmetry algebras and an infinite number of conserved quantities for the equations.

Arkadiev, Pogrebkov and Polivanov [3] (1989) studied the solutions of the Davey-Stewartson II equations whose singularities form closed lines with string-like behavior. They [4] (1989) also applied the inverse sacttering transform method to the Davey-Stewartson II equations. We refer [1] for more information on the Davey-Stewartson equations in terms of inverse scattering and integrable systems. Gilson and Nimmo [8] (1991) found dromion solutions and Malanyuk [11, 12] (1991, 1994) obtained finite-gap solutions of the equations. van de Linden [15] (1992) studied the solutions under a certain boundary condition. Clarkson and Hood [5] (1994) obtained certain symmetry reductions of the equations to ordinary differential equations with no intervening steps and provided new exact solutions which are not obtainable by the Lie group approach. Guil and Manas [9] (1995) found certains solutions of the Davey-Stewartson I equations by deforming dromion. Manas and Santini [10] (1997) studied a large class of solutions of the Davey-Stewartson II equations by a Wronskian scheme. It is obvious that the some of above solutions are equivalent to each other under the known symmetric transformations. It is time to study solutions of the Davey-Stewartson equations modulo the known symmetric transformations.

We observed in [17] that the argument functions of all the solutions of the two-dimensional cubic nonlinear Schrödinger equation in [15] are quadratic in spacial variables. This motivated us [17] to introduce a quadratic-argument approach to study exact solutions of the two-dimensional cubic nonlinear Schrödinger equation and the coupled two-dimensional cubic nonlinear Schrödinger equations modulo the known symmetry transformations. Indeed, our solution sets are most complete among the ones whose argument functions are quadratic in spacial variables. In this paper, we use the quadratic-argument approach to study exact solutions of the Davey-Stewartson equations modulo the most known symmetry transformations.

For convenience, we always assume that all the involved partial derivatives of related functions always exist and we can change orders of taking partial derivatives. We also use prime $'$ to denote the derivative of any one-variable function. The known symmetry transformations we are concerned with are:

$$\mathcal{T}_1(u) = e^{-(\epsilon_1\alpha' x+\beta' y+\gamma)i}\xi(t, x+\alpha, y+\beta), \tag{1.3}$$

$$\mathcal{T}_1(v) = v(t, x+\alpha, y+\beta) + \epsilon_1\alpha'' x + \beta'' y - \frac{\epsilon_1(\alpha')^2+(\beta')^2}{2} + \gamma'; \tag{1.4}$$

$$\mathcal{T}_2(u) = b^{-1}u(b^2t, bx, by), \qquad \mathcal{T}_2(v) = b^{-2}v(b^2t, bx, by); \tag{1.5}$$

where α, β, γ are functions of t and b is a nonzero real constant. The above transformations transform one solution of (1.1) and (1.2) into another solution. In particular, applying the transformation $\mathcal{T}_1$ to any known solution would yield solutions with three additional parameter functions.

2 Exact Solutions

Write

$$u = \xi(t, x, y)e^{i\phi(t,x,y)}, \tag{2.1}$$

where ξ and ϕ are real functions in t, x, y. Note

$$u_t = (\xi_t + i\xi\phi_t)e^{i\phi}, \qquad u_x = (\xi_x + i\xi\phi_x)e^{i\phi}, \qquad u_y = (\xi_y + i\xi\phi_y)e^{i\phi}, \tag{2.2}$$

$$u_{xx} = (\xi_{xx} - \xi\phi_x^2 + i(2\xi_x\phi_x + \xi\phi_{xx}))e^{i\phi}, \;\; u_{yy} = (\xi_{yy} - \xi\phi_y^2 + i(2\xi_y\phi_y + \xi\phi_{yy}))e^{i\phi}. \tag{2.3}$$

Then (1.1) is equivalent to

$$\begin{aligned} &2i\xi_t - 2\xi\phi_t + \epsilon_1(\xi_{xx} - \xi\phi_x^2 + i(2\xi_x\phi_x + \xi\phi_{xx})) \\ &+\xi_{yy} - \xi\phi_y^2 + i(2\xi_y\phi_y + \xi\phi_{yy}) - 2\epsilon_2\xi^3 - 2\xi v = 0, \end{aligned} \tag{2.4}$$

equivalently,

$$2\xi_t + 2(\epsilon_1\xi_x\phi_x + \xi_y\phi_y) + \xi(\epsilon_1\phi_{xx} + \phi_{yy}) = 0, \tag{2.5}$$

$$\xi(2\phi_t + \epsilon_1\phi_x^2 + \phi_y^2) - \epsilon_1\xi_{xx} - \xi_{yy} + 2\epsilon_2\xi^3 + 2\xi v = 0. \tag{2.6}$$

Moreover, (1.2) becomes

$$v_{xx} - \epsilon_1(v_{yy} + 2(\xi^2)_{xx}) = 0. \tag{2.7}$$

In order to solve the above system of partial differential equations, we assume in this section that

$$\phi = \alpha' x^2 + \beta' y^2 \tag{2.8}$$

for some functions α and β of t. Then (2.5) becomes

$$\xi_t + 2(\epsilon_1\alpha' x\xi_x + \beta' y\xi_y) + (\epsilon_1\alpha' + \beta')\xi = 0. \tag{2.9}$$

Thus

$$\xi = e^{-\epsilon_1\alpha - \beta}\vartheta(\varpi_1, \varpi_2), \qquad \varpi_1 = e^{-2\epsilon_1\alpha}x, \; \varpi_2 = e^{-2\beta}y, \tag{2.10}$$

where ϑ is some two-variable function in ϖ_1 and ϖ_2. Moreover, (2.6) is equivalent to

$$\begin{aligned} &2((\alpha'' + 2\epsilon_1(\alpha')^2)x^2 + (\beta'' + 2(\beta')^2)y^2)\vartheta - \epsilon_1 e^{-4\epsilon_1\alpha}\vartheta_{\varpi_1\varpi_1} \\ &-e^{-4\beta}\vartheta_{\varpi_2\varpi_2} + 2\epsilon_2 e^{-2\epsilon_1\alpha - 2\beta}\vartheta^3 + 2\vartheta v = 0. \end{aligned} \tag{2.11}$$

Case 1. $\vartheta = a\varpi_1 + b\varpi_2 + c$ for $a, b, c \in \mathbb{R}$.

In this case, The above equation is equivalent to

$$(\alpha'' + 2\epsilon_1(\alpha')^2)x^2 + (\beta'' + 2(\beta')^2)y^2 + \epsilon_2 e^{-2\epsilon_1\alpha-2\beta}(ae^{-2\epsilon_1\alpha}x + be^{-2\beta}y + c)^2 + v = 0. \quad (2.12)$$

So

$$\begin{aligned} v &= -[\alpha'' + 2\epsilon_1(\alpha')^2 + \epsilon_2 a^2 e^{-6\epsilon_1\alpha-2\beta}]x^2 - [\beta'' + 2(\beta')^2 + \epsilon_2 b^2 e^{-2\epsilon_1\alpha-6\beta}]y^2 \\ &\quad -2ab\epsilon_2 e^{-4\epsilon_1\alpha-4\beta}xy - \epsilon_2 c e^{-2\epsilon_1\alpha-2\beta}(2ae^{-2\epsilon_1\alpha}x + 2be^{-2\beta}y + c). \end{aligned} \quad (2.13)$$

Moreover, (2.7) is equivalent to

$$\alpha'' + 2\epsilon_1(\alpha')^2 + \epsilon_2 a^2 e^{-6\epsilon_1\alpha-2\beta} = \epsilon_1[\beta'' + 2(\beta')^2 + \epsilon_2 b^2 e^{-2\epsilon_1\alpha-6\beta}]. \quad (2.14)$$

To solve the above equation, we write

$$\alpha = \Im + \epsilon_1\beta. \quad (2.15)$$

Then the above equation becomes

$$\Im'' + 2\epsilon_1(\Im')^2 + 4\Im'\beta' + \epsilon_2(a^2 e^{-6\epsilon_1\Im} - \epsilon_1 b^2 e^{-2\epsilon_1\Im})e^{-8\beta} = 0. \quad (2.16)$$

Assuming $ab \neq 0$ and $\epsilon_1 = 1$, we have the following simple solution

$$\Im = \frac{1}{2}(\ln|a| - \ln|b|), \quad (2.17)$$

where β is any function of t. Suppose $a = b = 0$. Then we take

$$\beta' = -\frac{\Im''}{4\Im'} - \frac{\epsilon_1\Im'}{2} \Longrightarrow \beta = -\frac{1}{4}\ln\Im' - \frac{\epsilon_1\Im}{2} \quad (2.18)$$

modulo the transformation in (1.5), where $\Im$ is arbitrary increasing function of t. Then

$$\alpha = (\Im')^{-\epsilon_1/4}e^{\Im/2}, \qquad \beta = (\Im')^{-/4}e^{-\epsilon_1\Im/2}. \quad (2.19)$$

By (2.8), (2.10) and (2.13), we have:

Theorem 2.1. *Let $\Im$ be any increasing function of t and let $a, b, c \in \mathbb{R}$ with $ab \neq 0$. Suppose that β is any function of t. We have the following solutions of the Davey-Stewartson equations (1.1) and (1.2):*

$$\begin{aligned} u &= c\exp(-\epsilon_1(\Im')^{-\epsilon_1/4}e^{\Im/2} - (\Im')^{-/4}e^{-\epsilon_1\Im/2}) \\ &\quad \times \exp\frac{(2(\Im')^2 - \epsilon_1\Im'')x^2 - (2\epsilon_1(\Im')^2 + \Im'')y^2}{4\Im'}i, \end{aligned} \quad (2.20)$$

$$
\begin{aligned}
v &= \frac{\epsilon_1(\Im''' - (\Im'')^2) - 2(\Im')^2\Im''}{4(\Im')^2}x^2 + \frac{\Im''' - (\Im'')^2 + 2\epsilon_1(\Im')^2\Im''}{4(\Im')^2}y^2 \\
&\quad -\epsilon_2 c^2 \exp(-2\epsilon_1(\Im')^{-\epsilon_1/4}e^{\Im/2} - 2(\Im')^{-/4}e^{-\epsilon_1\Im/2}); \qquad (2.21)
\end{aligned}
$$

if $\epsilon_1 = 1$,

$$
u = e^{\beta'(x^2+y^2)i-(2\beta+(\ln|a|-\ln|b|)/2)}(ae^{\ln|b|-\ln|a|-2\beta}x + be^{-2\beta}y + c), \qquad (2.22)
$$

$$
\begin{aligned}
v &= -[\beta'' + 2(\beta')^2 + \epsilon_2|a|^{-1}|b|^3e^{-8\beta}](x^2+y^2) - 2a^{-1}b^3\epsilon_2 e^{-8\beta}xy \\
&\quad -\epsilon_2 c e^{-4\beta}(2a^{-1}b^2e^{-2\beta}x + 2|a|^{-1}b|b|e^{-2\beta}y + c). \qquad (2.23)
\end{aligned}
$$

Remark 2.2. Applying the symmetry transformation $\mathcal{T}_1$ in (1.3) and (1.4) to the above solutions, we get the following solutions with additional three parameter functions $\alpha_1, \beta_1, \gamma_1$ of t:

$$
\begin{aligned}
u &= c\exp\left(\frac{(2(\Im')^2 - \epsilon_1\Im'')(x+\alpha_1)^2 - (2\epsilon_1(\Im')^2 + \Im'')(y+\beta_1)^2}{4\Im'} - \epsilon_1\alpha_1'x - \beta_1'y - \gamma_1\right)i \\
&\quad \times \exp(-\epsilon_1(\Im')^{-\epsilon_1/4}e^{\Im/2} - (\Im')^{-/4}e^{-\epsilon_1\Im/2}),, \qquad (2.24)
\end{aligned}
$$

$$
\begin{aligned}
v = \frac{\epsilon_1(\Im''' - (\Im'')^2) - 2(\Im')^2\Im''}{4(\Im')^2}(x+\alpha_1)^2 + \frac{\Im''' - (\Im'')^2 + 2\epsilon_1(\Im')^2\Im''}{4(\Im')^2}(y+\beta_1)^2 + \epsilon_1{\alpha_1'}'x \\
+{\beta_1'}'y - \epsilon_2 c^2\exp(-2\epsilon_1(\Im')^{-\epsilon_1/4}e^{\Im/2} - 2(\Im')^{-/4}e^{-\epsilon_1\Im/2}) - \frac{\epsilon_1(\alpha_1')^2 + (\beta_1')^2}{2} + \gamma_1'; \qquad (2.25)
\end{aligned}
$$

if $\epsilon_1 = 1$,

$$
\begin{aligned}
u &= e^{[\beta'(x^2+y^2)-\epsilon_1\alpha_1'x+\beta_1'y+\gamma_1]i-(2\beta+(\ln|a|-\ln|b|)/2)} \\
&\quad \times(ae^{\ln|b|-\ln|a|-2\beta}(x+\alpha_1) + be^{-2\beta}(y+\beta_1) + c), \qquad (2.26)
\end{aligned}
$$

$$
\begin{aligned}
v &= -[\beta'' + 2(\beta')^2 + \epsilon_2|a|^{-1}|b|^3e^{-8\beta}]((x+\alpha_1)^2 + (y+\beta_1)^2) + \epsilon_1{\alpha_1'}'x + {\beta_1'}'y \\
&\quad -\epsilon_2 c e^{-4\beta}(2a^{-1}b^2e^{-2\beta}(x+\alpha_1) + 2|a|^{-1}b|b|e^{-2\beta}(y+\beta_1) + c) \\
&\quad -2a^{-1}b^3\epsilon_2 e^{-8\beta}(x+\alpha_1)(y+\beta_1) - \frac{\epsilon_1(\alpha_1')^2 + (\beta_1')^2}{2} + \gamma_1'. \qquad (2.27)
\end{aligned}
$$

Case 2. $\epsilon_1\alpha = \beta$.

We set

$$
\zeta_1(s) = \sinh s, \quad \eta_1(s) = \cosh s, \quad \zeta_{-1}(s) = \sin s, \quad \eta_{-1}(s) = \cos s. \qquad (2.28)
$$

Let ℓ and ℓ_1 be two fixed real constants. We denote

$$\varpi = \zeta_{\epsilon_1}(\ell)\varpi_1 + \eta_{\epsilon_1}(\ell)\varpi_2 + \ell_1. \tag{2.29}$$

Moreover, we assume

$$\vartheta = \nu(\varpi). \tag{2.26}$$

Observe

$$(\partial_x^2 - \epsilon_1\partial_y^2)(\nu(\varpi)) = -\epsilon_1\nu''(\varpi) \tag{2.30}$$

and

$$(\epsilon_1\partial_x^2 + \partial_y^2)(\nu(\varpi)) = (\eta_{\epsilon_1}(2\ell))\nu''(\varpi). \tag{2.31}$$

By (2.7), we further assume

$$v = \gamma(\epsilon_1 x^2 + y^2) + e^{-4\beta}(c + \zeta_{\epsilon_1}^2(\ell)\nu^2) \tag{2.32}$$

for a function γ of t and a real constant c. So (2.7) naturally holds.

Now (2.11) becomes

$$2(\beta'' + 2(\beta')^2 + \gamma)(\epsilon_1 x^2 + y^2)\nu + e^{-4\beta}(c\nu - \eta_{\epsilon_1}(2\ell)\nu'' + 2(\epsilon_2 + \zeta_{\epsilon_1}^2(\ell))\nu^3) = 0. \tag{2.33}$$

For simplicity, we take

$$\gamma = -\beta'' - 2(\beta')^2. \tag{2.34}$$

So (2.29) is equivalent to

$$c\nu - \eta_{\epsilon_1}(2\ell)\nu'' + 2(\epsilon_2 + \zeta_{\epsilon_1}^2(\ell))\nu^3 = 0. \tag{2.35}$$

First we have simple solution

$$\nu = \frac{1}{\varpi}\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2 + \zeta_{\epsilon_1}^2(\ell)}}, \qquad c = 0 \tag{2.36}$$

Recall

$$(\tan s)'' = 2(\tan^3 s + \tan s), \qquad (\sec s)'' = 2\sec^3 s - \sec s, \tag{2.37}$$

$$(\coth s)'' = 2(\coth^3 s - \coth s), \qquad (\mathrm{csch}\, s)'' = 2\mathrm{csch}^3 s + \mathrm{csch}\, s. \tag{2.38}$$

Denote Jacobi elliptic functions

$$\mathrm{sn}\, s = \mathrm{sn}\,(s|m), \qquad \mathrm{cn}\, s = \mathrm{cn}\,(s|m), \qquad \mathrm{dn}\, s = \mathrm{dn}\,(s|m), \tag{2.39}$$

where m is the elliptic modulus (e.g., cf. [16]). Then

$$(\mathrm{sn}\, s)'' = 2m^2\mathrm{sn}^3 s - (1+m^2)\mathrm{sn}\, s, \tag{2.40}$$

$$(\mathrm{cn}\ s)'' = -2m^2\mathrm{cn}^{\,2}s + (2m^2-1)\mathrm{cn}\ s, \tag{2.41}$$

$$(\mathrm{dn}\ s)'' = -2\mathrm{dn}^{\,3}s + (2-m^2)\mathrm{dn}\ s. \tag{2.42}$$

Moreover,

$$\lim_{m\to 1}\mathrm{sn}\ s = \tanh s, \qquad \lim_{m\to 1}\mathrm{cn}\ s = \lim_{m\to 1}\mathrm{dn}\ s = \mathrm{sech}\ s. \tag{2.43}$$

Comparing (2.35) with the equations in (2.37), (2.38) and (2.39)-(2.42), we have the following solutions:

$$\nu = \sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ \tan\varpi, \qquad c = 2\eta_{\epsilon_1}(2\ell); \tag{2.44}$$

$$\nu = \sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ \sec\varpi, \qquad c = -\eta_{\epsilon_1}(2\ell); \tag{2.45}$$

$$\nu = \sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ \coth\varpi, \qquad c = -2\eta_{\epsilon_1}(2\ell); \tag{2.46}$$

$$\nu = \sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ \mathrm{csch}\ \varpi, \qquad c = \eta_{\epsilon_1}(2\ell); \tag{2.47}$$

$$\nu = m\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ \mathrm{sn}\ \varpi, \qquad c = -(1+m^2)\eta_{\epsilon_1}(2\ell); \tag{2.48}$$

$$\nu = m\sqrt{\frac{-\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ \mathrm{cn}\ \varpi, \qquad c = (2m^2-1)\eta_{\epsilon_1}(2\ell), \tag{2.49}$$

$$\nu = \sqrt{\frac{-\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ \mathrm{dn}\ \varpi, \qquad c = (2-m^2)\eta_{\epsilon_1}(2\ell). \tag{2.50}$$

By (2.8), (2.10) and (2.32), we have:

Theorem 2.3. *Let $\ell, \ell_1, m \in \mathbb{R}$. Suppose that β is any function of t. Take the notations in (2.28). The followings are solutions of the Davey-Stewartson equations (1.1) and (1.2) (where the solution exists only when the expression makes sense as real function):*

$$u = \frac{e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}}{e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1}\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}, \tag{2.51}$$

$$v = -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2) + \frac{e^{-4\beta}\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{(\epsilon_2+\zeta_{\epsilon_1}^2(\ell))(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)^2}; \tag{2.52}$$

$$u = \sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\ e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}\tan(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1), \tag{2.53}$$

$$\begin{aligned} v &= e^{-4\beta}\left(\frac{\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\tan^2(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)+2\eta_{\epsilon_1}(2\ell)\right) \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2); \end{aligned} \tag{2.54}$$

$$u=\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\, e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}\sec(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1), \tag{2.55}$$

$$\begin{aligned} v &= e^{-4\beta}\left(\frac{\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\sec^2(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)-\eta_{\epsilon_1}(2\ell)\right) \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2); \end{aligned} \tag{2.56}$$

$$u=\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\, e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}\coth(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1), \tag{2.57}$$

$$\begin{aligned} v &= e^{-4\beta}\left(\frac{\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\coth^2(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)-2\eta_{\epsilon_1}(2\ell)\right) \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2); \end{aligned} \tag{2.58}$$

$$u=\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\, e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}\mathrm{csch}\,(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1), \tag{2.59}$$

$$\begin{aligned} v &= e^{-4\beta}\left(\frac{\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\mathrm{csch}^{\,2}(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)+\eta_{\epsilon_1}(2\ell)\right) \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2); \end{aligned} \tag{2.60}$$

$$u=m\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\, e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}\mathrm{sn}\,(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1), \tag{2.61}$$

$$\begin{aligned} v &= e^{-4\beta}\left(\frac{m^2\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\mathrm{sn}^{\,2}(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)-(1+m^2)\eta_{\epsilon_1}(2\ell)\right) \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2); \end{aligned} \tag{2.62}$$

$$u=m\sqrt{\frac{-\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\, e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}\mathrm{cn}\,(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1), \tag{2.53}$$

$$\begin{aligned} v &= e^{-4\beta}\left((2m^2-1)\eta_{\epsilon_1}(2\ell)-\frac{m^2\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\mathrm{cn}^{\,2}(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)\right) \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2); \end{aligned} \tag{2.64}$$

$$u=\sqrt{\frac{-\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\, e^{-2\beta+(\epsilon_1x^2+y^2)\beta' i}\mathrm{dn}\,(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1), \tag{2.65}$$

$$\begin{aligned} v &= e^{-4\beta}\left((2-m^2)\eta_{\epsilon_1}(2\ell)-\frac{m^2\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\mathrm{dn}^{\,2}(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)x+\eta_{\epsilon_1}(\ell)y)+\ell_1)\right) \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1x^2+y^2); \end{aligned} \tag{2.66}$$

Remark 2.4. Applying the symmetry transformation $\mathcal{T}_1$ in (1.3) and (1.4) to the above solutions, we can obtain solutions with additional three parameter functions. For instance, we get the following solutions with additional three parameter functions $\alpha_1, \beta_1, \gamma_1$ of t from the above first two solutions:

$$u = \frac{e^{-2\beta+[(\epsilon_1(x+\alpha_1)^2+(y+\beta_1)^2)\beta'-\epsilon_1\alpha_1'x-\beta_1'y-\gamma_1]i}}{e^{-2\beta}(\zeta_{\epsilon_1}(\ell)(x+\alpha_1)+\eta_{\epsilon_1}(\ell)(y+\beta_1))+\ell_1}\sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}, \tag{2.67}$$

$$\begin{aligned} v &= -(\beta''+2(\beta')^2)(\epsilon_1(x+\alpha_1)^2+(y+\beta_1)^2)+\epsilon_1\alpha_1''x+\beta_1''y-\frac{\epsilon_1(\alpha_1')^2+(\beta_1')^2}{2}+\gamma_1 \\ &\quad +\frac{e^{-4\beta}\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{(\epsilon_2+\zeta_{\epsilon_1}^2(\ell))(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)(x+\alpha_1)+\eta_{\epsilon_1}(\ell)(y+\beta_1))+\ell_1)^2}; \end{aligned} \tag{2.68}$$

$$\begin{aligned} u &= \sqrt{\frac{\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}}\, e^{-2\beta+[(\epsilon_1(x+\alpha_1)^2+(y+\beta_1)^2)\beta'-\epsilon_1\alpha_1'x-\beta_1'y-\gamma_1]i} \\ &\quad \times \tan(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)(x+\alpha_1)+\eta_{\epsilon_1}(\ell)(y+\beta_1))+\ell_1), \end{aligned} \tag{2.69}$$

$$\begin{aligned} v &= e^{-4\beta}\left(\frac{\zeta_{\epsilon_1}^2(\ell)\eta_{\epsilon_1}(2\ell)}{\epsilon_2+\zeta_{\epsilon_1}^2(\ell)}\tan^2(e^{-2\beta}(\zeta_{\epsilon_1}(\ell)(x+\alpha_1)+\eta_{\epsilon_1}(\ell)(y+\beta_1))+\ell_1)+2\eta_{\epsilon_1}(2\ell)\right)+\gamma_1' \\ &\quad -(\beta''+2(\beta')^2)(\epsilon_1(x+\alpha_1)^2+(y+\beta_1)^2)+\epsilon_1\alpha_1''x+\beta_1''y-\frac{\epsilon_1(\alpha_1')^2+(\beta_1')^2}{2}. \end{aligned} \tag{2.70}$$

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